Perfect matchings in 3-partite 3-uniform hypergraphs

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Abstract

Let H be a 3-partite 3-uniform hypergraph with each partition class of size n, that is, a 3-uniform hypergraph such that every edge intersects every partition class in exactly one vertex. We determine the Dirac-type vertex degree thresholds for perfect matchings in 3-partite 3-uniform hypergraphs.

Keywords: hypegraph, k-partite, perfect matching, minimum degree

1 Introduction

A perfect matching in a graph G is a set of vertex disjoint edges, which covers all vertices of G. Tutte [18] gave a characterisation on all those graphs that contain a perfect matching. An easy consequence of a celebrated theorem of Dirac [6] shows that if G is a graph of order n even and the minimum degree $\delta(G) \geq n/2$, then G contains a perfect matching. Thus, it is natural to ask for Dirac-type degree thresholds for perfect matchings in hypergraphs.

We follow the notation of [3]. For a set M and an integer k, denote by $\binom{M}{k}$ the set of all k-set of M. A k-uniform hypergraph, or k-graph for short, is a pair $H = (V(H), E(H))$, where $V(H)$ is a finite set of vertices and $E(H) \subset \binom{V(H)}{k}$ is a set of k-sets of $V(H)$. Often we write V instead of $V(H)$ when it is clear from the context. A matching M in H is a set of vertex disjoint edges of H, and it is perfect if M covers all vertices of H. Clearly, a perfect matching only exist if $|V|$ is divisible by k.

For a k-graph H and an l-set $T \in \binom{V}{l}$, let $\deg(T)$ be the number of $(k-l)$-sets $S \in \binom{V}{k-l}$ such that $S \cup T$ is an edge in H, and let $\delta_l(H)$ be the minimum l-degree of H, that is, $\min \deg(T)$ over all $T \in \binom{V}{l}$. We define $m_l(k,n)$ to be the smallest integer m such that every k-graph H of order n satisfying $\delta_l(H) \geq m$ contains a perfect matching. Hence, we always assume that $k|n$ whenever we talk about $m_l(k,n)$. Thus, we have $m_1(2,n) = n/2$.

For $k \geq 2$ and $l = k-1$, Rödl, Ruciński, Szemerédi [17] determined the value of $m_{k-1}(k,n)$ exactly, which improved the bounds given in [11, 16]. For $k \geq 3$ and $1 \leq l < k$, it is conjectured in [8] that

$$m_l(k,n) \sim \max\left\{\frac{1}{2}, 1 - \left(1 - \frac{1}{k}\right)^{k-l}\right\} \binom{n}{k-l}. \tag{1}$$

For $k = 3$ and $l = 1$, Hàn, Person and Schacht [8] showed that (1) is true, that is, $m_1(3,n) \sim \frac{5}{9}\binom{n}{2}$ improving the result of Daykin and Häggkvist [5] for $r = 3$. The exact value was independently determined by Khan [9] and Kühn, Osthus and Treglown [12]. Khan [10] further determined $m_1(4,n)$ exactly. For $k \geq 3$ and $k/2 \leq l < k$, Pikhurko [14] proved that $m_l(k,n) \sim \frac{1}{2}\binom{n}{k-l}$. Thus, for $1 \leq l < k/2$, (1) is still open except for the two cases $k = 3$ and $l = 1$, and $k = 4$ and $l = 1$. Partial results were proved by Hàn, Person and Schacht [8], later improved by the first author and Ruciński [13]. We recommend [15] for a survey for other results about perfect matchings in hypergraphs.

Instead of seeking for a perfect matching, Bollobás, Daykin, Erdős [4] considered the Dirac-type degree thresholds for a matching of size d.

Theorem 1.1 (Bollobás, Daykin, Erdős [4]). *Let $k \geq 2$ be an integer. If H be a k-graph H of order $n \geq 2k^2(d+2)$ and*

$$\delta_1(H) > \binom{n-1}{k-1} - \binom{n-d}{k-1},$$

then H contains a matching of size d.

For $k = 3$, Kühn, Osthus, and Treglown [12] extended the above result and proved that if $\delta_1(H) > \binom{n-1}{2} - \binom{n-d}{2}$, then H contains a matching of size $d \leq n/3$.

A k-graph H is k-partite, if there exists a partition of the vertex set V into k classes $V_1, \ldots, V_k$ such that every edge intersects every class in exactly one vertex. Clearly, a perfect matching only exist if $|V_1| = |V_2| = \cdots = |V_k|$. For a k-partite k-graph H and integer $1 \leq l \leq k$, a l-set $T \in \binom{V}{l}$ is said to be legal if $|T \cap V_i| \leq 1$ for $1 \leq i \leq k$, and let $\delta_l(H) = \min \deg(T)$ over all legal l-sets in H. Similarly, we define $m'_l(k,n)$ to be the smallest integer m such that every k-partite k-graph H with each n vertices in each class satisfying $\delta_l(H) \geq m$ contains a perfect matching. Note that we no longer assume that $k|n$ for $m'_l(k,n)$. Aharoni, Georgakopoulos and Sprüssel [1] proved that $m'_{k-1}(k,n) \leq n/2+1$. Pikhurko [14] proved an Ore-type result for perfect matchings in k-partite k-graphs. For $k \in \mathbb{N}$, we refer to the set $\{1, 2, \ldots k\}$ as $[k]$. For a set $L \in \binom{[k]}{l}$, a set $T \in \binom{V}{l}$ an L-tuple if $|T \cap V_i| = 1$ for $i \in L$, and let $\delta_L(H) = \min \deg(T)$ overall L-tuple T. Thus, the precise statement of the Ore-type result proved by Pikhurko is as follows.

Theorem 1.2 (Pikhurko [14]). *Let $k \geq 2$, $l < k$ and $L \in \binom{[k]}{l}$ be fixed. Let H be a k-partite k-graph with partition classes $V_1, \ldots, V_k$ each of size n. If*

$$\frac{\delta_L(H)}{n^{k-l}} + \frac{\delta_{[k]\setminus L}(H)}{n^l} \geq 1 + o(1)$$

then H contains a perfect matching.

This implies that for $k/2 \leq l < k$, $m'_l(k,n) \sim \frac{1}{2}n^{k-l}$. In this paper, we determine $m'_1(3,n)$ that is the minimum $\delta_1(H)$ that ensures a perfect matching in 3-partite 3-graphs H.

Theorem 1.3. *There is an n_0 such that if H is a 3-partite 3-graph with $n > n_0$ vertices in each class and*

$$\delta_1(H) > \begin{cases} (5n^2 - 6n)/9 & \text{if } n = 0 \pmod 3, \\ (5n^2 - 4n - 1)/9 & \text{if } n = 1 \pmod 3, \\ (5n^2 - 8n + 5)/9 & \text{if } n = 2 \pmod 3, \end{cases}$$

then H contains a perfect matching.

Note that these bounds are optimal by considering $H_3(n;n-1)$ and $H_3^*(n;n-1)$ defined in Example 2.1 and Example 2.2 respectively. We make no attempt to optimise the value of n_0 in the above theorem. A rough calculation shows that $n_0 = 2^{672}$ suffices. It is clear that n_0 can be reduced significantly by using more careful counting arguments in various places of the proof. Let H be a 3-partite 3-graph on 6 vertices $u_1, u_2, v_1, v_2, w_1, w_2$ with four edges $u_1v_1w_1$, $u_1v_2w_2$, $u_2v_2w_1$ and $u_2v_1w_2$. Observe that $\delta_1(H) = 2$ and H does not contain a perfect matching. Hence, $n_0 > 2$ and we ask whether Theorem 1.3 is true for $n_0 = 3$.

In addition, we also prove a natural generalisation of Theorem 1.1 for k-partite k-graphs H, that is, a Dirac-type $\delta_1(H)$ threshold for a matching of size m in k-partite k-graphs H.

Theorem 1.4. *Let k, m and n be integers such that $k \geq 2$ and $n \geq k^7 m$. Let H be a k-partite k-graph with n vertices in each class. Suppose the largest matching in H is of size m and*

$$\delta_1(H) \geq (m - \lceil m/k \rceil) n^{k-2}.$$

Then, $H \subset H_k(n;m)$ if $m \neq 1 \pmod{k}$, and $H \subset H_k(n;m-1) \cup H'$ if $m \neq 1 \pmod{k}$, where the graph $H_k(n;m)$ is defined in Example 2.1 and $E(H')$ is an intersecting family.

2 Notations and an outline of the proof of Theorem 1.3

For $a, b \in \mathbb{N}$, we refer to the set $\{a, a+1, \dots b\}$ as $[a,b]$. For $k|m$, we say that a vertex set W is a *balanced m-set* if $|W \cap V_i| = m/k$ for $i \in [k]$. Throughout this paper, unless stated otherwise, $v_{i,j} = V_i \cap V(e_j)$ for partition class V_i and edges e_j. Next, we give two examples to show that the bounds stated in Theorem 1.3 are best possible.

Example 2.1 ($H_k(n;m)$)**.** *For $i \in [k]$, let $d_i \leq n$ and let U_i and W_i be a partition of vertex set of size n with $|W_i| = d_i$. Let $U = \bigcup_{i\in[k]} U_i$ and $W = \bigcup_{i\in[k]} W_i$. Define $H(n;d_1,\dots,d_k)$ to be the k-partite k-graph with partition classes $U_1 \cup W_1, \dots, U_k \cup W_k$ consisting of all those edges which meet W. Thus, $H(n;d_1,\dots,d_k)$ has a matching of size $\min\{n, \sum_{i\in[k]} d_i\}$. Write $H_k(n;m)$ for $H(n;d_1,\dots,d_k)$, where $d_i = \lfloor (m+i-1)/k \rfloor$ for $i \in [k]$. Note that for $l \leq k$*

$$\delta_l(H_k(n;m)) = n^{k-l} - \prod_{i\in[k-l]} \left(n - \left\lfloor \frac{m+i-1}{k} \right\rfloor \right).$$

In particular, for $k = 3$, $l = 1$ and $m = n-1$ we have

$$\delta_1(H_3(n;n-1)) = \begin{cases} (5n^2 - 6n)/9 & \text{if } n = 0 \pmod{3}, \\ (5n^2 - 4n - 1)/9 & \text{if } n = 1 \pmod{3}, \\ (5n^2 - 8n - 5)/9 & \text{if } n = 2 \pmod{3}, \end{cases}$$

so for $n \neq 2 \pmod{3}$ the bound in Theorem 1.3 is best possible.

Example 2.2 ($H_k^*(n;m)$)**.** *For a k-partite k-graph H, the edge set $E(H)$ is an intersecting family if $V(e) \cap V(e') \neq \emptyset$ for $e, e' \in E(H)$. Given $H_k(n;m-1)$, we fix $u_i \in U_i$ for $i \in [k]$ and let S be the set of all legal k-sets T such that $|T \cap \{v_1, \dots, v_k\}| > k/2$. Define $H_k^*(n;m)$ to be the k-partite*

k*-graph on vertex set* $V(H_k(n; m-1))$ *with* $E(H_k^*(n; m)) = E(H_k(n; m-1)) \cup S$*. Note that* $H_k^*(n; m)$ *has matchings of sizes at most* m *and*

$$\delta_1(H_k^*(n; m)) = \delta_1(H_k(n; m-1)) + \sum_{k/2 < i \leq k-1} \binom{k-1}{i} n^{k-i-1}.$$

In particular, for $k = 3$ *and* $m = n-1$ *we have* $\delta_1(H_3^*(n; n-1)) = (5n^2 - 8n + 4)/9$*. Therefore, for* $n = 2 \pmod 3$ *the bound in Theorem 1.3 is best possible.*

Now, we set up notations for 3-partite 3-graphs H with partition classes, V_1, V_2, V_3, each of size n. Given three vertex sets U_1, U_2, U_3, not necessarily disjoint, we say an edge $u_1u_2u_3$ is of *type* $U_1U_2U_3$ if $u_i \in U_i$ for $i \in [3]$.

For $i \in [3]$, let $d_i \leq n$ and define $H'(n; d_1, d_2, d_3)$ to be the resulting subgraph of $H(n; d_1, d_2, d_3)$ after removing all edges of type WWW. In other words, $H'(n; d_1, d_2, d_3)$ is the 3-partite 3-graph with partition classes $U_1 \cup W_1$, $U_2 \cup W_2$, $U_3 \cup W_3$ consisting of all those edges of type UUW and UWW, where $U = U_1 \cup U_2 \cup U_3$, $W = W_1 \cup W_2 \cup W_3$, $|W_i| = d_i$ and $|U_i| = n - d_i$ for $i \in [3]$. Define $H_3'(n; m)$ analogously.

For $\epsilon > 0$, we say that H is ϵ*-close to* $H'(n; d_1, d_2, d_3)$ if

$$|E(H'(n; d_1, d_2, d_3)) \backslash E(H)| \leq \epsilon n^3.$$

Denote by L_x the *link graph* of $x \in V$ consisting of all 2-element sets T such that $\{x\} \cup T$ is an edge in H. Given disjoint vertex sets $U_1, \dots, U_s$, let $L_x(U_1, \dots, U_s)$ be the set of all 2-element sets $T \subset \bigcup_{j \in [s]} U_i \cap L_x$ with $|T \cap U_j| \leq 1$ for $j \in [s]$. For a matching $M = \{e_1, \dots, e_s\}$, we write $L_x(M)$ or $L_x(e_1, \dots, e_s)$ for $L_x(V(e_1), \dots, V(e_s))$.

We now give an outline of the proof of Theorem 1.3. Suppose that H is not ϵ-close to $H_3'(n; n)$. We remove a small matching M satisfying the conditions in the absorption lemma, Lemma 4.2, and call the resulting graph H'. This matching M has the property that for every 'small' balanced set $W \subset V(H')$ there exists a perfect matching in $H[V(M) \cup W]$. If we can show that H' contains a matching covering most of its vertices, then the remaining vertices can be absorbed into M to get a perfect matching in H. Rather than getting bolt down by the details, we first prove an asymptotic version of Theorem 1.3, Theorem 5.2, in Section 5 to setup the framework. We then refine these arguments in Section 6. The case when H is ϵ-close to $H_3'(n; n)$ is dealt with in Section 7 by the application of Theorem 1.4, which is in turn proven in Section 3.

3 Partial matchings

We are going to prove the analogue of Theorem 1.1 for k-partite k-graphs H, Theorem 1.4. First, we prove the case for $n \geq k^4 m^2$.

Lemma 3.1. *Let* k*,* m *and* n *be integers such that* $k \geq 2$ *and* $n \geq k^4 m^2/2$*. Let* H *be a* k*-partite* k*-graph with* n *vertices in each class. Suppose the largest matching in* H *is of size* m *and* $\delta_1(H) \geq (m - \lceil m/k \rceil) n^{k-2}$*. Then,* $H \subset H_k(n; m)$ *if* $m \neq 1 \pmod k$*, and* $H \subset H_k(n; m-1) \cup H'$ *if* $m \neq 1 \pmod k$*, where* $E(H')$ *is an intersecting family.*

Proof. For $k = 2$, the theorem can be easily verified by König's theorem and the minimum degree condition. Also it is trivial for $m = 1$, so we may

assume that $k \geq 3$ and $m \geq 2$. Let $V_1, \ldots, V_k$ be the partition classes of H and let M be a largest matching in H. Write $m = rk + s$ with $r \geq 0$ and $1 \leq s \leq k$, so $\delta_1(H) \geq (m-r-1)n^{k-2}$. Let $X_i = V_i \backslash V(M)$ for $i \in [k]$ and $X = \bigcup_{i \in [k]} X_i$. For $x \in X$, the number of edges containing x and exactly one vertex of M is at least

$$\delta_1(H) - \binom{k-1}{2} m^2 n^{k-3} \geq \left(m - r - 1 - k^{-2}\right) n^{k-2}. \tag{2}$$

Given $i \in [k]$, $x \in X$ and $e_j \in M$, we say that x and e_j are i-connected if there exist more than $(2k)^{k-2}$ edges containing both x and $v_{i,j}$ but no other vertices in M. In addition, we say that x makes l connections to a submatching $M' subset M$ if there exists l pairs $(e_j, i) \in M' \times [k]$ such that x and e_j are i-connected. Each $x \in X$ makes at least $m - r - 1$ connections to M, or else by (2) we have

$$(m - r - 1 - k^{-2})n^{k-2} \leq mk \times (2k)^{k-2} + (m - r - 2)n^{k-2}$$

a contradiction. Note that if there exists $e_j \in M$ such that e_j and x are i-connected and e_j and $x' \in X$ are i'-connected for some $i \neq i'$ and distinct $x, x' \in X$, then we can enlarge M by replacing e_j with two disjoint edges in $H[X \cup V(e_j)]$ containing the vertex pair $(x, v_{i,j})$ and $(x', v_{i',j})$ respectively. Thus, if e_j and x are i-connected and e_j and x' are i'-connected, then $i = i'$ or $x = x'$. Also notice that if $x \in X$ is i-connected to $e_j \in M$ for two values of i, then e_j is not i'-connected to any $x' \in X \backslash \{x\}$ for any $i' \in [k]$. Thus, there are at most $m - 1$ vertices $x \in X$ that have two connections with some $e \in M$ and we call these vertices *bad*.

First we consider the case when $s \neq 1$. We now claim that there exists a partition of M into k classes $M_1, \ldots, M_k$ such that if $x \in X$ and $e \in M_j$ are i-connected, then $i = j$. First, pick $x_k \in X_k$ that is not bad. For $i \in [k]$, define M'_i to be the set of edges $e \in M$ such that x_k and e are i-connected. Note that $M'_k = \emptyset$. Since x_k makes at least $m - r - 1$ connections to M, $|M'_1| + \cdots + |M'_{k-2}| \geq m - r - 1$. Recall that $m \geq rk + 2$, we may assume that $|M'_1| \geq r + 1$. Pick $x_1 \in X_1$ that is not bad. For $i \in [k]$, let M_i to be the set of edges $e \in M$ such that $e \in M'_i$ or x_1 and e are i-connected. Since there are no connection between x_1 and M_1, we have

$$|M_1| + \sum_{2 \leq i \leq k} |M_i| \geq r + 1 + (m - r - 1) = m$$

and so $M_1, \ldots, M_k$ forms a partition of M. In addition, since equality holds, we have $|M_i| \leq r + 1$. Without loss of generality, we have $|M_i| = r + 1$ for $i \in [s]$ and $|M_i| = r$ otherwise. Moreover for $i \in [s]$ and $x \in X_i$, x is j-connected to $e \in M_j$ for $j \neq i$. Set $W_i = V(M_i) \cap V_i$ for $i \in [k]$ and $W = \bigcup_{i \in [k]} W_i$. In order to show that $H \subset H_k(n, m)$, it is sufficient to show that all edges meet W. Suppose the contrary and let e' be an edge disjoint from W. Clearly e' must intersect with some $e \in M$ by the maximality of M. Let $e_1, \ldots, e_l$ be the edges in M that intersect with e' and $e_j \in M_{i_j}$. Next we pick vertices $x_1, \ldots, x_l \in X \backslash V(e')$ such that $x_j \in X_1$ if $i_j \neq 1$ otherwise $x_j \in X_2$. Note that x_j and e_j are i_j-connected for $j \in [l]$. Hence, there exist disjoint edges $e'_1, \ldots, e'_l$ in $V \backslash V(e' \cup e_1 \cup \cdots \cup e_l)$ with $\{x_j, v_{i_j,j}\} \subset e'_j$ for $j \in [l]$. Therefore, $M \cup \{e', e'_1, \ldots, e'_l\} \backslash \{e_1, \ldots, e_l\}$ is a matching of size $m+1$, a contradiction.

Now, we consider the case when $s = 1$. For $i \in [k]$, define M_i to be the set of edges $e \in M$ such that x and e are i-connected for at least $2k$

vertices $x \in X$. By similar arguments as in the case when $2 \le s \le l$ and averaging argument, we have $\sum_{i\in[k]} |M_i| \ge m-1$ and $r \le |M_i| \le r+1$. Set $W_i = V(M_i) \cap V_i$ for $i \in [k]$ and $W = \bigcup_{i\in[k]} W_i$. If $M = \bigcup_{i\in[k]} M_i$, then we can deduce that every edges meet W and so $H \subset H_k(n;m)$ (by using the argument in the previous paragraph). Thus, we may assume that $e_0 \in M\backslash \bigcup_{i\in[k]} M_i$. Consider all the edges that do not meet W. Suppose there exists two disjoint such edges e_1' and e_2'. Let $e_1, \ldots, e_l$ be the edges in $M\backslash\{e_0\}$ that intersect with e_1' or e_2'. Note that we can enlarge M by replacing each e_j with an edge containing x_j and $v_{i_j,j}$, where $e_j \in M_{i_j}$ and x_j are i_j-connected to e_j. Thus, we conclude that the set of all the edges that do not meet $\bigcup_{i\in[k]} W_i$ forms an intersecting family, so $H \subset H_k(n;m-1) \cup H'$ such that $E(H')$ is an intersecting family. Hence, the proof of the lemma is completed. □

To prove Theorem 1.4 for the case $k^7 m \le n \le k^4 m^2/2$, we proceed by induction on m. Its proof is based on [4]. First, we need the following simple proposition of which we omit the proof.

Proposition 3.2. *Let H be a k-partite k-graph with n vertices in each class. Suppose the largest matching in H is of size m.*

(a) *If $H\backslash v$ contains a matching of size m for some vertex v, then*

$$\deg(v) \le n^{k-1} - (n-m)^{k-1}.$$

(b) *For a legal k-set T, if $H\backslash T$ contains a matching of size $m-k+1$, then for some $v \in T$*

$$\deg(v) \le n^{k-1} - (n-m)^{k-1} \le kmn^{k-2}.$$

Proof of Theorem 1.4. For $k = 2$, the theorem can be easily verified by König's theorem and the minimum degree condition. We may assume that $k \ge 3$. Write $m = rk + s$ with $r \ge 0$ and $1 \le s \le k$, so $\delta_1(H) \ge (m-r-1)n^{k-2}$. Fix s and we proceed by induction on r. For $r \le k^2$, the theorem is true by Lemma 3.1. Suppose that $r > k^2$ and the results holds for smaller values of r. Let $V_1, \ldots, V_k$ be the partition classes of H and let M be a matching of size m in H. In the next claim, we show that there is a vertex $v_i \in V_i$ with $\deg(v_i) \ge n^{k-1}/2k^2$ for $i \in [k]$. The proof of the claim uses the ideas in the proof of Lemma 3.1.

Claim 3.3. *For $i \in [k]$, there exists $v_i \in V_i$ such that $\deg(v_i) \ge n^{k-1}/2k^2$.*

Proof. Let $X_i = V_i\backslash V(M)$ for $i \in [k]$ and $X = \bigcup_{i\in[k]} X_i$. For $x \in X$, the number of edges containing x and exactly one vertex of M is at least

$$\delta_1(H) - \binom{k-1}{2} m^2 n^{k-3} \ge \left(\frac{k-1}{k} - \frac{1}{k^3} - \frac{1}{k^4}\right) mn^{k-2}. \tag{3}$$

Given $x \in X$ and $e_j \in M$, we say that x and e_j are i-connected if there exist more than $(2k)^{k-2}$ edges containing both x and $v_{i,j}$ but no other vertices in M. Again, we say that x makes l connections to a submatching $M' subset M$ if there exists l pairs $(e_j, i) \in M' \times [k]$ such that x and e_j are i-connected. By (3), we know that that $x \in X$ makes at least $\left(\frac{k-1}{k} - \frac{1}{k^3} - \frac{2}{k^4}\right) m$ connections to M. In addition, if $e_j \in M$ and $x \in X$ are i-connected and e_j and $x' \in X$ are i'-connected, then $i = i'$ or $x = x'$. Thus, there are at most $m-1$ vertices $x \in X$ that have two connections with some $e \in M$ and we remove those vertices from X.

For $i \in [k]$, define M_i to be the set of edges $e \in M$ such that x and e are i-connected for at least $2k$ vertices $x \in X$. We are going to show that M_1, $\ldots$, M_k cover all but at most m/k^2 edges $e \in M$. Pick m vertices $x_1, \ldots,$ $x_m \in X_k$. Since each x_j makes at least $\left(\frac{k-1}{k} - \frac{1}{k^3} - \frac{2}{k^4}\right) m$ connections to M, by averaging $|M_1|+\cdots+|M_{k-1}| \geq \left(\frac{k-1}{k} - \frac{1}{k^3} - \frac{3}{k^4}\right) m$ and similarly we have

$$\sum_{j\in[k]} |M_j| - |M_i| \geq \left(\frac{k-1}{k} - \frac{1}{k^3} - \frac{3}{k^4}\right) m$$

for $i \in [k]$. Hence, we have

$$\sum_{j\in[k]} |M_j| \geq \left(1 - \frac{1}{k^2(k-1)} - \frac{3}{k^3(k-1)}\right) m \geq \left(1 - \frac{1}{k^2}\right) m$$

and $|M_i| \leq (1/k+1/k^3+3/k^4)$ for $i \in [k]$. Set $W_i = V(M_i)\cap V_i$ for $i \in [k]$ and set $W = \bigcup_{i\in[k]} W_i$. Further, set $M_0 = M\setminus \bigcup M_i$, so $|M_0| \leq m/k^2$. As in the proof of Lemma 3.1, we can deduce that every edge e in $H[V\setminus M_0]$ must intersect with W.

For each $x \in X_2$, by (3) the number of edges containing x and exactly one vertex of W_1 is at least

$$\begin{aligned}&\left(\frac{k-1}{k} - \frac{1}{k^3} - \frac{1}{k^4}\right) mn^{k-2} - \sum_{3\leq i\leq k} |M_i|n^k - |M_0| \left(n^{k-2} + (k-1)(2k)^k\right) \\ \geq& mn^{k-2}/k^2.\end{aligned}$$

Thus, there exists $v_1 \in W_1$ such that $\deg(v_1) \geq n^{k-1}/2k^2$ and similarly there exists $v_i \in W_i$ such that $\deg(v_i) \geq n^{k-1}/2k^2$ for $i \in [k]$. Therefore, the proof of the claim is completed. $\square$

Let $T = \{v_1, \ldots, v_k\}$ with $v_i \in V_i$ and $\deg(v_i) \geq n^{k-1}/2k^2$, and let $H' = H\setminus T$. Then

$$\begin{aligned}\delta_1(H') \geq& (m-r-1)n^{k-2} - (n^{k-1} - (n-1)^{k-1}) \\ \geq& (m-k-r)(n-1)^{k-2}.\end{aligned}$$

For $s \neq 1$, if there does not exist a matching of size greater than $m-k$ in H', then the induction hypothesis implies that $H' \subset H_k(n-1; m-k)$ and so $H \subset H_k(n;m)$ and the corresponding statement holds for $s = 1$. Thus we can assume without loss of generality that H' contains a matching of size $m-k+1$. By Claim 3.3 and Proposition 3.2 (b), there exists an $i \in [k]$ such that

$$n^{k-1}/2k^2 < \deg(v_i) \leq kmn^{k-2}$$

which implies that $n < 4k^3m$, a contradiction. $\square$

Corollary 3.4. *Let $r \geq 0$ and $1 \leq s \leq 3$ be integers and let*

$$d_3(n,r,s) = \begin{cases} n^2 - (n-r)^2 + 1 & \text{if } s = 1, \\ n^2 - (n-r)(n-r-1) & \text{if } s = 2, \\ n^2 - (n-r-1)^2 & \text{if } s = 3. \end{cases}$$

Let H be a 3-partite 3-graph with each class of size $n \geq 3^7m$ and $\delta_1(H) > d_3(n,r,s)$. Then, H contains a matching of size $3r+s+1$.

Proof. The case when $2 \le s \le 3$ follows easily from Theorem 1.4 and Example 2.1. To prove the case when $s = 1$, it is enough to show that if the edge set of a 3-partite 3-graph H' forms an intersecting family, then $\delta_1(H') \le 1$. Assume that $\delta_1(H') \ge 2$. Since $\delta_1(H') \ge 1$, it is easy to see that there exist $e_1, e_2 \in E(H')$ with $|V(e_1) \cap V(e_2)| = 1$, say $e_1 = v_{1,1}v_{2,1}u$ and $e_1 = v_{1,2}v_{2,2}u$. Let $w \in V_3\backslash\{u\}$. If $e_3 = v_{1,1}v_{2,2}w$ and $e_4 = v_{1,2}v_{2,1}w$ exist, then $\{e_1, e_2, e_3, e_4, e\}$ is not an intersecting family for edges e with $V(e)\cap\{u, w\} = \emptyset$. Hence, we may assume that $e_3 = v_{1,1}v_{2,2}w$ and $e_5 = v_{1,1}v_{2,1}w$ exist. However, $\{e_1, e_2, e_3, e_5, e\}$ is not an intersecting family for edges e with $V(e) \cap \{v_{1,1}, v_{1,2}\} = \emptyset$, a contradiction. □

4 An absorption lemma for k-partite k-graphs

Here, we prove a version of the absorption lemma given by Hàn, Person, Schacht [8] for k-partite k-graphs. Thus, its proof follows the same arguments as in [8]. For the sake of completeness, we include the proof below. First we need the following simple proposition.

Proposition 4.1. *Let H be a k-partite k-graph with n vertices in each class. For all $x \in [0, 1]$ and all integers $1 \le m \le l \le k$ we have, if $\delta_l(H) \ge xn^{k-l}$, then $\delta_m(H) \ge xn^{k-m}$.*

Proof. Let $V_1, \ldots, V_k$ be the vertex classes of H. Let T be a balanced m-set in $V(H)$. Without loss of generality, $T = \{v_1, \ldots, v_m\}$ with $v_i \in V_i$ for $\in [m]$. Then the condition on δ_l implies that T is contained in at least

$$\binom{k-m}{l-m}^{-1} \sum_{m<i_{m+1}<\cdots<i_l\le k} \sum_{v_{m+1}\in V_{i_{m+1}}} \cdots \sum_{v_l \in V_{i_l}} deg(v_1, \ldots, v_l)$$
$$\ge \binom{k-m}{l-m}^{-1} \binom{k-m}{l-m} n^{l-m} xn^{k-l} = xn^{k-m}$$

edges, and the proposition follows. □

Lemma 4.2 (An absorption lemma for k-partite k-graphs)**.** *Let $k > l > 0$,*

$$0 < \gamma < \frac{(k-1)^{k^2-2k-2}}{5ke^{k(k-1)}} \text{ and } \gamma' = \gamma^{2k}k(k-1)^2/2.$$

Then, there is an n_0 such that for all $n > n_0$ there following holds: Suppose H is a k-partite k-graph with n vertices in each class and minimum l-degree $\delta_l(H) \ge (1/2 + \gamma)n^{k-l}$, then there exists a matching M in H of size $|M| \le \gamma^k n$ such that for every balanced set W of size $|W| \le k\gamma' n$, there exists a matching covering exactly the vertices of $V(M) \cup W$.

Proof. Let H be a k-partite k-graph with partition classes $V_1, \ldots, V_k$ each of size n and minimum l-degree $\delta_l(H) \ge (1/2+\gamma)n^{k-l}$. From Proposition 4.1, $\delta_1(H) \ge (1/2 + \gamma)n^{k-1}$ and it suffices to prove the lemma for $l = 1$. Throughout the proof we may assume that n_0 is chosen sufficiently large. Furthermore set $m = k(k-1)$ and call a balanced m-set A an *absorbing* m-set for a balanced k-set T if A spans a matching of size $k-1$ and $A \cup T$ spans a matching of size k, in other words, $A \cap T = \emptyset$ and both $H[A]$ and $H[A \cup T]$ contain a perfect matching. Denote by $\mathcal{L}(T)$ the set

of all absorbing m-sets for T. Next, we show that for every balanced k-set T, there are many absorbing m-sets for T.

Claim 4.3. *For every balanced k-set T, $|\mathcal{L}(T)| \geq \gamma^{k-1} n^m / 2(k-1)^k$.*

Proof. Let $T = \{v_1, \ldots, v_k\}$ be fixed with $v_i \in V_i$ for $i \in [k]$. Since n_0 was chosen large enough, there are at most $(k-1)n^{k-2} \leq \gamma n^{k-1}$ edges, which contain v_1 and v_j for some $j \in [2, k]$. Due to the minimum degree of H there are at least $n^{k-1}/2$ edges containing v_1 but none of $v_2, \ldots, v_k$. We fix one such edge $\{v_1, u_2, \ldots, u_k\}$ with $u_i \in V_i$ for $i \in [2, k]$. Set $U_1 = \{u_2, \ldots, u_k\}$ and $W_0 = T$. For each $j \in [2, k]$ and each pair u_j, v_j suppose we succeed to choose a $(k-1)$-set U_j such that U_j is disjoint to $W_{j-1} = U_{j-1} \cup W_{j-2}$ and both $U_j \cup \{u_j\}$ and $U_j \cup \{v_j\}$ are edges in H. Then for a fixed $j \in [2, k]$ we call such a choice U_j *good*, motivated by $A = \bigcup_{j \in [k]} U_j$ being an absorbing m-set for T.

Note that in each step $j \in [2, k]$ there are $k + (j-1)(k-1)$ vertices in W_{j-1}. More specifically, there are at most $j \leq k$ vertices in $V_i \cap W_{j-1}$ for $i \in [k]$. Thus, the number of edges intersecting u_j (or v_j respectively) and at least one other vertex in W_j is at most $(k-1)jn^{k-2} < k^2 n^{k-2} \leq \gamma n^{k-1}$. For each $j \in [2, k]$ there are at least $2\gamma n^{k-1} - \gamma n^{k-1} = \gamma n^{k-1}$ choices for U_j and in total we obtain $\gamma^{k-1} n^{k(k-1)}/2$ absorbing m-sets for T with multiplicity at most $(k-1)^k$. $\square$

Now, choose a family F of balanced m-sets by selecting each of the $\binom{n}{k-1}^k$ possible balanced m-sets independently with probability $p = \gamma^k n / \Delta$ with

$$\Delta = 2\left(\frac{ne}{k-1}\right)^m \geq 2\binom{n}{k-1}^k. \tag{4}$$

Then, by Chernoff's bound (see e.g. [2]) with probability $1 - o(1)$ as $n \to \infty$, the family F satisfies the following properties:

$$|F| \leq \gamma^k n \tag{5}$$

and

$$|\mathcal{L}(T) \cap F| \geq \frac{\gamma^{2k-1}(k-1)^{k(k-2)} n}{5e^m} \geq 2\gamma' n \tag{6}$$

for all balanced k-sets T. Furthermore, we can bound the expected number of intersecting m-sets by

$$\binom{n}{k-1}^k \times k(k-1) \times \binom{n}{k-2}\binom{n}{k-1}^{k-1} \times p^2 \leq \gamma' n$$

Thus, using Markov's inequality, we derive that with probability at least 1/2

$$F \text{ contains at most } \gamma' n \text{ intersecting pairs.} \tag{7}$$

Hence, with positive probability the family F has all properties stated in (5), (6) and (7). By deleting all the intersecting balanced m-sets and non-absorbing m-sets in such a family F, we get a subfamily F' consisting of pairwise disjoint balanced m-sets, which satisfies

$$|\mathcal{L}(T) \cap F'| \geq 2\gamma' n - \gamma' n \geq \gamma' n \tag{8}$$

for all balanced k-sets T. Since F' consists only of absorbing m-sets, $H[V(F')]$ has a perfect matching M of size at most $\gamma^k n$. For a balanced set $W \subset V \backslash V(M)$ of size $|W| \leq k\gamma' n$, W can be partition in to at most $k\gamma' n$ balanced k-set. Each balanced k-set can be successively absorbed using a different absorbing m-set in F', so there exists a matching covering $V(M) \cup W$. $\square$

5 Asymptotic result

In this section, we prove an asymptotic version of Theorem 1.3. Its proof will provide the framework for the proof of the exact result. We will need to following lemma.

Lemma 5.1. *Let $0 < \gamma < 3/4$ and $\rho > 0$. Let H be a 3-partite 3-graph with $n > \max\{3 \times 2^{27}/(\rho\gamma^6), 3^5 \times 2^{32}/(\rho\gamma^5)\}$ vertices in each class and*

$$\delta_1(H) \geq \left(\frac{5}{9} + \gamma\right) n^2,$$

then H contains a matching of size at least $(1-\rho)n$.

Before presenting its proof, we prove the asymptotic version of Theorem 1.3 using the above lemma and the absorption lemma, Lemma 4.2.

Theorem 5.2. *There exists a constant $c > 0$ such that for all $\gamma > 0$, if H is a 3-partite 3-graph with $n > c\gamma^{-12}$ vertices in each class and*

$$\delta_1(H) \geq \left(\frac{5}{9} + \gamma\right) n^2,$$

then H contains a perfect matching.

Proof. Without loss of generality, we may assume that c is sufficiently large and $\gamma < 2e^{-6}/15$ satisfying the hypothesis of Lemma 4.2 with $k = 3$ and $l = 1$. Let M be the matching given by Lemma 4.2 and so $|M| \leq \gamma^3 n$. Let $H' = H \backslash V(M)$. Note that

$$\delta_1(H) \geq \left(\frac{5}{9} + \gamma\right) n^2 - 2\gamma^3 n^2 \geq \left(\frac{5}{9} + \gamma/2\right) n'^2,$$

where $n' = n - |M|$. By Lemma 5.1 taking $\rho = 6\gamma^6$, there exists a matching M' in H' of size at least $(1 - 6\gamma^6)n'$. Let $W = V(H') \backslash V(M')$. Note that W is balanced with at most $6\gamma^6 n' < 6\gamma^6 n$ vertices. Thus, there exists a matching M'' covering exactly the vertices of $V(M') \cup W$ by Lemma 4.2, and so M' and M'' form a perfect matching. $\square$

In order to prove Lemma 5.1, we will need the following simple fact for graphs of which we omit its proof.

Proposition 5.3. *Let G be a graph of order n. If $e(G) > 2\epsilon\binom{n}{2}$, then there exists a subgraph $G' \subset G$ with $\delta(G') \geq \epsilon n$ and $e(G \backslash G') < 2\epsilon\binom{n}{2}$.*

Proof of Lemma 5.1. Without loss of generality, we may assume that $\gamma < 10^{-3}$ and n is sufficiently large. Let V_1, V_2 and V_3 be the partition classes in H and let M be a largest matching in H. Assume to the contrary that $n - |M| \geq \rho n$. Let $X_i = V_i \backslash V(M)$ for $i \in [3]$ and $X = X_1 \times X_2 \times X_3$. Without loss of generality, we may suppose that $x = |X_1| = |X_2| = |X_3| = \rho n$ (we omit floors and ceilings for clarity of presentation).

For every $S = (x_1, x_2, x_3) \in X$ and any submatching M' of M, let $L_S(M') = L_{x_1}(M') \cup L_{x_2}(M') \cup L_{x_3}(M')$. Note that for each $i \in [3]$, $L_{x_i}(M')$ only has edges between V_{i+1} and V_{i+2} (addition modulo 3), so $L_S(M')$ does not contain any multiple edges. For $S \in X$ and $E \in \binom{M}{2}$, we say that E is of *type* (a_1, a_2, a_3) *with respect to* S if $e(L_{x_i}(E)) = a_i$ for $i \in [3]$. Note that if $v_{i_1,j_1}v_{i_2,j_2} \in L_S(e_1, e_2)$ for $e_1, e_2 \in M$, then $i_1 \neq i_2$ and $j_1 \neq j_2$. Thus, $0 \leq a_1, a_2, a_3 \leq 2$.

For every $S \in X$, define the *matching graph* G_S *with respect to* S to be an edge-coloured complete graph with vertex set M and for $e, e' \in M$ ee' is coloured (a_1, a_2, a_3) if $\{e, e'\}$ is of type (a_1, a_2, a_3) with respect to S. Let $G_S(a_1, a_2, a_3)$ be the subgraph of G_S induced by edges of colour (a_1, a_2, a_3) and write $e_S(a_1, a_2, a_3) = |E(G_S(a_1, a_2, a_3))|$.

First, we show that $L_S(E)$ does not contain a matching of size 3 for $S \in X$ and $E \in \binom{M}{2}$.

Claim 5.4. *For all $S \in X$ and all $E \in \binom{M}{2}$, $L_S(E)$ does not contain a matching of size 3. Moreover, $e_S(a_1, a_2, a_3) = 0$ for $a_1 + a_2 + a_3 \geq 5$ and $S \in X$.*

Proof. Suppose that there exist $S \in X$ and $E \in \binom{M}{2}$ such that $L_S(E)$ contains a matching M_0 of size 3, say $M_0 = \{T_1, T_2, T_3\}$. Since each partition class of $L_S(E)$ is of size 2, we may assume that $T_i \cap V_i = \emptyset$ for $i \in [3]$. But then the sets $\{x_i\} \cup T_i$, $i \in [3]$ form a matching in H of size 3 which intersects only 2 edges of M. This is a contradiction with the maximality of M in H. The last assertion is easily follows as $v_{i_1,j_1}v_{i_2,j_2} \notin L_S(e_1, e_2)$ for $i_1 = i_2$ or $j_1 = j_2$. $\square$

In the following claim, we show that at most $x^3/8$ sets $S \in X$ satisfying $e_S(1, 2, 1) \geq \frac{\gamma}{2}\binom{m}{2}$.

Claim 5.5. *For all but at most $x^3/8$ sets $S \in X$, we have $e_S(1, 2, 1) < \frac{\gamma}{2}\binom{m}{2}$. Moreover, the above statement still holds if we replace $e_S(1, 2, 1)$ by $e_S(2, 1, 1)$ or $e_S(1, 1, 2)$.*

Proof. Suppose that the claim is false, so there are at least $x^3/4$ sets $S \in X$ with $e_S(1, 2, 1) \geq \frac{\gamma}{2}\binom{m}{2}$. Fix one such S. By Claim 5.4, note that $E = (e_1, e_2) \in E_S(1, 2, 1)$ if and only if $L_S(E)$ is a path of length 4. To be more precise, $L_E(S) = v_{1,2}v_{3,1}v_{2,2}v_{1,1}v_{3,2}$ or $v_{1,1}v_{3,2}v_{2,1}v_{1,2}v_{3,1}$. We write $\overrightarrow{e_1e_2}$ if $L_E(S) = v_{1,2}v_{3,1}v_{2,2}v_{1,1}v_{3,2}$. Thus, S induces an orientation on $G_S(1, 2, 1)$.

By Proposition 5.3, there exists a subgraph $G' \subset G_S(1, 2, 1)$ such that $\delta(G) \geq \gamma m/8$ and $e(G \backslash G') < \frac{\gamma}{4}\binom{m}{2}$, so $|G'| \geq \gamma m/8$. Thus, there are at least $(\gamma m/8)^6/4 = \gamma^6 m^6/2^{20}$ copies of a path of length 5 in $G' \subset G_S(1, 2, 1)$. Then by averaging, there exists a copy $P_0 = e_1e_2e_3e_4e_5e_6$ of a path of length 5 such that $P_0 \subset G_S(1, 2, 1)$ for at least $\gamma^6 x^3/2^{22}$ sets $S \in X$. In addition, there are at least $\gamma^6 x^3/2^{27}$ sets $S \in X$ such that every such S induces the same orientation on P_0. Since $\gamma^6 x^3/2^{27} > 3x^2$, there exist four such sets $S_1, \ldots, S_4 \in X$ that are vertex disjoint. In order to get a contradiction with the maximality of M, we are going to show that there exists a matching M' of size $r + 1$ in $L_{S_i}(e_1, \ldots, e_6)$ for $i \in [4]$ touching at most r of the e_i. Moreover, M' combined with $S_1, \ldots, S_4$ will yield an enlargement of M.

Without loss of generality, we may assume that $\overleftarrow{e_3e_4}$. If $\overrightarrow{e_1e_2}$, then we may assume that $\overrightarrow{e_2e_3}$ by restricting to $e_1e_2e_3e_4$. Note that there exists a matching of size 5 in $L_{S_i}(e_1, e_2, e_3, e_4)$ for $i \in [4]$, namely, $\{v_{1,1}v_{2,2},$

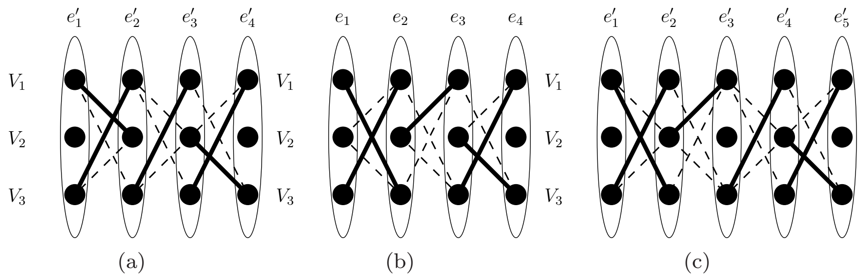


Figure 1: Diagrams for Claim 5.5

$v_{3,1}v_{1,2},\ v_{3,2}v_{1,3},\ v_{3,3}v_{1,4},\ v_{2,3}v_{3,4}\}$ (see Figure 1 (a)), and so we obtain a contradiction. Thus, we have $\overleftarrow{e_1e_2}$.

If $\overleftarrow{e_2e_3}$, then there exists a matching of size 5 in $L_{S_i}(e_1, e_2, e_3, e_4)$ for $i \in [4]$, namely, $\{v_{1,1}v_{3,2},\ v_{3,1}v_{2,2},\ v_{1,2}v_{2,3},\ v_{1,3}v_{3,4},\ v_{3,3}v_{1,4}\}$ (see Figure 1 (b)). Thus, we have $\overrightarrow{e_2e_3}$. By similar argument as in the previous paragraph, we deduce that $\overrightarrow{e_4e_5}$ by considering $e_2e_3e_4e_5$.

In summary, we have $\overrightarrow{e_1e_2}$, $\overleftarrow{e_2e_3}$, $\overrightarrow{e_3e_4}$ and $\overleftarrow{e_4e_5}$. Note that there is a matching of size 6 in $L_{S_1}(e_1, e_2, e_3, e_4, e_5)$, namely, $\{v_{1,1}v_{3,2},\ v_{3,1}v_{1,2},\ v_{2,2}v_{1,3},\ v_{3,3}v_{1,4},\ v_{3,4}v_{1,5},\ v_{2,4}v_{3,5}\}$ (see Figure 1 (c)). Again, this matching together with $S_1, \dots, S_4$ will yield an enlargement of M, a contradiction. $\square$

Define $G'_S(a_1, a_2, a_3)$ to be a subgraph of $G_S(a_1, a_2, a_3)$ such that

$$\delta(G'_S(a_1, a_2, a_3)) \geq \gamma m/16$$

if exists. Furthermore, by Proposition 5.3, we may assume that

$$e(G_S(a_1, a_2, a_3) \backslash G'_S(a_1, a_2, a_3)) \leq \frac{\gamma}{8}\binom{m}{2}.$$

Let $G'_S(\cdot, \cdot, 0) = G'_S(2, 2, 0) \cup G'_S(2, 1, 0) \cup G'_S(1, 2, 0)$ and let $e'_S(\cdot, \cdot, 0) = |E(G'_S(\cdot, \cdot, 0))|$. Thus,

$$e'_S(\cdot, \cdot, 0) \geq e_S(2, 2, 0) + e_S(2, 1, 0) + e_S(1, 2, 0) - \frac{3\gamma}{8}\binom{m}{2}. \qquad (9)$$

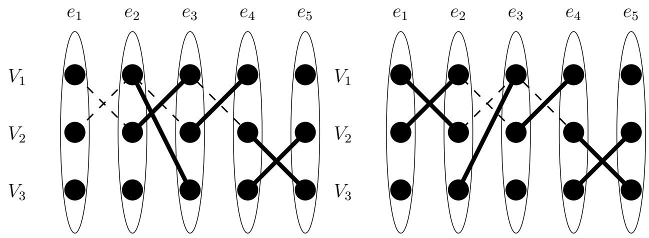
Claim 5.6. *For all but at most $x^3/8$ sets $S \in X$, no two of $\{G'_S(\cdot, \cdot, 0)$, $G'_S(\cdot, \cdot, 0)$, $G'_S(\cdot, \cdot, 0)\}$ intersect in more than $\gamma m/8$ vertices.*

Proof. Let A be the set of $(a_1, a_2, a_3, b_1, b_2, b_3)$ such that

$$(a_{\sigma(1)}, a_{\sigma(1)}, a_{\sigma(3)}, b_{\sigma(1)}, b_{\sigma(2)}, b_{\sigma(3)}) = (0, a'_2, 2, 2, b'_2, b'_3)$$

for some $1 \leq a'_2 \leq 2$, $0 \leq b'_2, b'_3 \leq 2$ and some permutation $\sigma \in S_3$. For $(a_1, a_2, a_3, b_1, b_2, b_3) \in \mathcal{A}$, denote by $\mathcal{P}(a_1, a_2, a_3, b_1, b_2, b_3)$ a path $e_1e_2e_3e_4e_5$ of length 4 such that e_ie_{i+1} has colour (a_1, a_2, a_3) for $i \in [3]$, and e_4e_5 has (b_1, b_2, b_3).

Suppose the statement in the claim is false, so there are at least $x^3/8$ such $S \in X$. Fix one such S. Without loss of generality, $|G'_S(0, \cdot, \cdot) \cap G'_S(\cdot, 0, \cdot)| \geq \gamma m/8$. Further, assume that $|G'_S(0, 2, 1) \cap G'_S(2, 0, 1)| \geq$



(a) (b)

Figure 2: Diagrams for Claim 5.6

$\gamma m/72$. Recall that $\delta(G'_S(0,2,1))$, $\delta(G'_S(2,0,1)) \geq \gamma m/16$. Thus, we can find at least

$$\frac{\gamma m}{72}\frac{\gamma m}{16}\left(\frac{\gamma m}{16}-1\right)\left(\frac{\gamma m}{16}-2\right)\left(\frac{\gamma m}{16}-3\right) \geq \frac{1}{9}\left(\frac{\gamma m}{16}\right)^5$$

copies of $\mathcal{P}(0,2,1,2,0,1)$ in G_S. Thus, for each such S there exist at least $(\gamma m/16)^5/9$ copies of $P(A)$ in S for some $A \in \mathcal{A}$.

Note that $|\mathcal{A}| \leq 108$. By averaging, we may assume that, for some $A \in \mathcal{A}$, there exists a copy $\mathcal{P}_0$ of $\mathcal{P}(A)$ such that $\mathcal{P}_0 \subset G_S$ for 3 disjoint sets $S_1, S_2, S_3 \in X$ such that $L(E) = L_{S_i}(E)$ for $i \in [3]$. Without loss of generality, assume that $A = (0,1,2,2,0,0)$. Hence, e_2e_3 is coloured $(0,1,2)$ and so either $v_{1,2}v_{3,3} \in L(e_2,e_3)$ or $v_{1,3}v_{3,2} \in L(e_2,e_3)$. If $v_{1,2}v_{3,3} \in L(e_2,e_3)$, then $\{v_{1,2}v_{3,3},\ v_{1,3}v_{2,2},\ v_{1,4}v_{2,3},\ v_{2,4}v_{3,5},\ v_{2,5}v_{3,4}\}$ (see Figure 2 (a)) is a matching of size 5 in $L(e_2,\dots,e_5)$. Notice that this matching spans 4 edges of M, and so by combining with S_1, S_2, S_3 we can obtain an enlargement of M, a contradiction. If $v_{1,3}v_{3,2} \in L(e_2,e_3)$, then $\{v_{1,1}v_{2,2},\ v_{2,1}v_{1,2},\ v_{1,3}v_{3,2},\ v_{1,4}v_{2,3},\ v_{2,4}v_{3,5},\ v_{2,5}v_{3,4}\}$ (see Figure 2 (b)) is a matching of size 6 in $L(e_1,\dots,e_5)$, which also implies a contradiction. $\square$

Let X' be the set of $S \in X$ such that

$$e_S(2,1,1), e_S(1,2,1), e_S(1,1,2) < \frac{\gamma}{8}\binom{m}{2}.$$

We are going to show that for every $S \in X'$,

$$|G'_S(0,\cdot,\cdot)|, |G'_S(\cdot,0,\cdot)|, |G'_S(\cdot,\cdot,0)| \geq \left(\frac{1}{3}+\frac{\gamma}{8}\right)m$$

and since $|X'| \geq 5x^3/8$ by Claim 5.5, it will contradict Claim 5.6. Fix $S = (x_1,x_2,x_3) \in X'$. Observe that the number of edges of H of the form $\{x\} \cup T$ and such that $T \notin L_x(M)$, is at most $2xm + m \leq \gamma n^2/2$. Hence, by the assumption on $\delta_1(H)$, for every $x \in X_i$,

$$e(L_x(M)) \geq \deg(x) - \frac{\gamma n^2}{2} \geq \left(\frac{5}{9}+\frac{\gamma}{2}\right)n^2 \geq \left(\frac{10}{9}+\gamma\right)\binom{m}{2}. \tag{10}$$

Let $G_S(a_1,*,*) = \bigcup_{0 \leq b_2,b_3 \leq 2} G_S(a_1,b_2,b_3)$ and write $e_S(a_1,*,*)$ for $|E(G_S(a_1,*,*))|$. Similarly, define $G_S(*,a_2,*)$, $G_S(*,*,a_3)$, $e_S(*,a_2,*)$ and $e_S(*,*,a_3)$.

Note that

$$e(L_{x_1}(M)) = 2e_S(2,*,*) + e_S(1,*,*) = e_S(2,*,*) + \binom{m}{2} - e_S(0,*,*)$$
$$\leq e_S(2,*,*) + \binom{m}{2} - e'_S(0,\cdot,\cdot) - e_S(0,2,0), \tag{11}$$

and so (10) becomes

$$e_S(2,*,*) \geq \left(\frac{1}{9}+\gamma\right)\binom{m}{2} + e'_S(0,\cdot,\cdot) + e_S(0,2,0). \tag{12}$$

On the other hand, by (9), we have

$$e_S(2,*,*) \leq e'_S(\cdot,\cdot,0) + e'_S(\cdot,0,\cdot) + e_S(2,1,1) + e_S(2,0,0) + \frac{3\gamma}{4}\binom{m}{2}$$
$$\leq e'_S(\cdot,\cdot,0) + e'_S(\cdot,0,\cdot) + e_S(2,0,0) + \frac{7\gamma}{8}\binom{m}{2}. \tag{13}$$

Thus, together with (12) we have

$$e'_S(\cdot,\cdot,0) + e'_S(\cdot,0,\cdot) + e_S(2,0,0) \geq \left(\frac{1}{9}+\frac{\gamma}{8}\right)\binom{m}{2} + e'_S(0,\cdot,\cdot) + e_S(0,2,0), \tag{14}$$

and by similar argument (i.e. swapping the indices) we have

$$e'_S(\cdot,\cdot,0) + e'_S(0,\cdot,\cdot) + e_S(0,2,0) \geq \left(\frac{1}{9}+\frac{\gamma}{8}\right)\binom{m}{2} + e'_S(\cdot,0,\cdot) + e_S(2,0,0) \tag{15}$$

Therefore, by adding (14) and (15) together we obtain

$$e'_S(\cdot,\cdot,0) \geq \left(\frac{1}{9}+\frac{\gamma}{8}\right)\binom{m}{2}. \tag{16}$$

Hence, $G'_S(\cdot,\cdot,0)$ spans at least $(1/3+\gamma/8)m$ vertices, and similar results hold for $G'_S(\cdot,0,\cdot)$ and $G'_S(0,\cdot,\cdot)$. This completes the proof of the lemma. $\square$

Notice that Claim 5.4, Claim 5.5 and Claim 5.6 do not require the assumption on $\delta_1(H)$. In fact, we have shown that if (10) holds, then for every $S \in X'$, then $|G'_S(0\cdot,\cdot)|, |G'_S(\cdot,0,\cdot)|, |G'_S(\cdot,\cdot,0)| \geq (1/3+\gamma/8)\,m$.

6 Extending Lemma 5.1

By extending the arguments in the proof of Lemma 5.1, we are going to show that if $\delta_1(H) \geq (5/9-\gamma)\,n^2$, then H contains a large matching or a large subgraph of order $3n'$ that is ϵ-close to $H'_3(n';n')$

Lemma 6.1. *Let $\gamma,\rho > 0$ and $\gamma'' = 3(1200\gamma + \sqrt{\gamma/2})$. There exists an n_0 such that if H is a 3-partite 3-graph with $n > n_0$ vertices in each class and $\delta_1(H) \geq (5/9-\gamma)\,n^2$ and contains no matching of size $(1-\rho)n$, then there exists a subgraph H' such that H' is $8\gamma''$-close to $H'_3(n';n')$, where $n' \geq (1-\gamma'')n$ and $3|n'$.*

Proof. Without loss of generality, we may assume that γ is small and n is sufficiently large. Let V_1, V_2 and V_3 be the partition classes of H and let M be a largest matching in H. Let $m = |M|$, so $n - m \geq \rho n$. Let $X_i = V_i \backslash V(M)$ for $i \in [3]$ and $X = X_1 \times X_2 \times X_3$. Thus $x = |X_i| \geq \rho n$. For every $S \in X$, define G_S, $G'_S(a_1, a_2, a_3)$, $G'_S(\cdot, \cdot, 0)$, $G'_S(a_1, *, *)$ as in the proof of Lemma 5.1. Next, we bound the number of edges of type $XXV(M)$.

Claim 6.2. *For $i \in [3]$, all but at most $x/8$ vertices in $x_i \in X_i$, we have $e(L_{x_i}(V(M), X)) \leq (1+\sqrt{\gamma})mx$, where $L_{x_i}(V(M), X) = L_{x_i}(V(M), X_1 \cup X_2 \cup X_3)$.*

Proof. Suppose the claim is false for $i = 1$ say. Let $x_1 \in X_1$ such that $e(L_{x_1}(V(M), X)) > (1 - \sqrt{\gamma})mx$. For an edge $e \in M$, we say that e is *good for* x_1 if each of $v_2 = e \cap V_2$ and $v_3 = e \cap V_3$ has degree at least 2 in $L_{x_1}(e, X)$. We claim that there are at least $2\sqrt{\gamma}m/3$ good edges for x_1. Indeed, otherwise we have

$$\begin{aligned} e(L_{x_1}(V(M), X) <&(1 - 2\sqrt{\gamma}/3)m(1 + x) + 4\sqrt{\gamma}mx/3 \\ =&(1 + 2\sqrt{\gamma}/3)mx + (1 - 2\sqrt{\gamma}/3)m \\ <&(1 + \sqrt{\gamma})mx, \end{aligned}$$

a contradiction. Since there are at least $x/8$ such $x_1 \in X_1$, by averaging there exists an edge $e_1 \in M$ that is good for both $x_1, x'_1 \in X_1$. It is easy to find $u \in X_3$ and $u' \in X_2$ such that $x_1 v_{2,1} u$ and $x'_1 u' v_{3,1}$ are disjoint edges in H. Hence, we can enlarge M contradicting the maximality of M. $\square$

Denote by X'' the set of $S = (x_1, x_2, x_3) \in X$ such that

(*a*) $e_S(2,1,1), e_S(1,2,1), e_S(1,1,2) \leq \frac{\gamma}{8}\binom{m}{2}$,

(*b*) $e(L_{x_i}(V(M), X)) \leq (1 + \sqrt{\gamma})mx$ for $i \in [3]$,

(*c*) no two of $\{G'(2, \cdot, \cdot), G'(\cdot, 2, \cdot), G'(\cdot, \cdot, 2)\}$ intersect in more than $\gamma m/8$ vertices.

Note that $X'' \subset X'$. Now, we show that $M \geq (1 - 3\sqrt{\gamma/2})n$.

Claim 6.3. *For $S = (x_1, x_2, x_3) \in X''$,*

$$L_{x_i}(M) \geq \left(\frac{10}{9} - 2\gamma\right)\binom{m}{2}$$

for $i \in [3]$. Moreover, $|M| \geq (1 - 3\sqrt{\gamma/2})n$.

Proof. For $i \in [3]$, the number of edges of type H of the form $\{x_i\} \cup T$ and such that $T \in \binom{V(M)}{2} \backslash L_{x_i}(M)$, is at most $m < \sqrt{\gamma}mx$. Hence

$$\begin{aligned} L_{x_i}(M) \geq&\delta_1(H) - e(L_{x_i}(V(M), X)) - \sqrt{\gamma}mx \\ \geq&(5/9 - \gamma)n^2 - (1 + 2\sqrt{\gamma})mx \\ =&(5/9 - \gamma)(m^2 + x^2) + (1/9 - 2\gamma - 2\sqrt{\gamma})mx \\ \geq&(5/9 - \gamma)(m^2 + x^2). \end{aligned} \tag{17}$$

Thus, the first inequality in the claim holds. Now suppose that $m \leq (1 - 3\sqrt{\gamma/2})n$ and so $x \geq 3\sqrt{\gamma/2}n \geq 3\sqrt{\gamma/2}m$. Hence, (17) becomes

$$L_{x_i}(M) \geq (5/9 - \gamma)\left(1 + \frac{9\gamma}{2}\right)m^2 \geq (10/9 + \gamma)\binom{m}{2}.$$

Recall that $S \in X'' \subset X'$. By the remark after the proof of Lemma 5.1, we can deduce that $|G'(2, \cdot, \cdots)|$, $|G'(\cdot, 2, \cdots)|$, $|G'(\cdot, \cdots, 2)| \geq (1+\gamma/8)m$ contradicting property (c). $\square$

By the remark after the proof of Lemma 5.1, Claim 5.4, Claim 5.5 and Claim 5.6 also hold. Thus together with Claim 6.2 and Claim 6.3, we know that $|X''| \geq x^3/8$. In the next claim, we show that given $S \in X''$ all but at most $126\gamma\binom{m}{2}$ edges of G_S have colours $(2,2,0)$, $(2,0,2)$, $(0,2,2)$ or $(1,1,1)$.

Claim 6.4. *For $S \in X''$,*

$$e'_S(2,2,0), e'_S(2,0,2), e'_S(0,2,2) \geq \left(\frac{1}{9} - 23\gamma\right)\binom{m}{2} \text{ and}$$

$$e_S(1,1,1) \geq \left(\frac{1}{9} - 57\gamma\right)\binom{m}{2}.$$

Proof. By Claim 6.3 and following the same argument as in the proof of Lemma 5.1, we obtain the following inequalities corresponding to (12), (13) and (16):

$$e_s(2,*,*) \geq \left(\frac{1}{9} - 2\gamma\right)\binom{m}{2} + e'_S(0,\cdot,\cdot) + e_s(0,2,0), \tag{18}$$

$$e_S(2,*,*) \leq e'_S(\cdot,\cdot,0) + e'_S(\cdot,0,\cdot) + e_S(2,0,0) + \frac{7\gamma}{8}\binom{m}{2}, \tag{19}$$

$$e'_S(\cdot,\cdot,0) \geq \left(\frac{1}{9} - 3\gamma\right)\binom{m}{2}, \tag{20}$$

and similar inequalities hold if we swap the indices. Hence,

$$(1/3 - 5\gamma)m \leq |G'_S(\cdot,\cdot,0)|, |G'_S(\cdot,0,\cdot)|, |G'_S(0,\cdot,\cdot)| \leq (1/3 + 11\gamma)m,$$

where the lower and upper bounds are due to (20) and property (c) respectively. Moreover,

$$e'_S(\cdot,\cdot,0) \leq \left(\frac{1}{9} + 8\gamma\right)\binom{m}{2}. \tag{21}$$

Denote by $Q_S(j)$ the set of $E \in \binom{M}{2}$ such that E is of type (a_1, a_2, a_3) with respect to S and $a_1 + a_2 + a_3 = j$. By Claim 5.4, $Q_S(j) = \emptyset$ for $j \geq 5$. In addition, we have

$$\begin{aligned}|Q_S(4)| =& e_S(2,2,0) + e_S(2,0,2) + e_S(0,2,2) \\ &+ e_S(2,1,1) + e_S(1,2,1) + e_S(1,2,1) \\ \leq& e'_S(2,2,0) + e'_S(2,0,2) + e'_S(0,2,2) + \frac{3\gamma}{4}\binom{m}{2}.\end{aligned}$$

Therefore,

$$e(L_S(M)) \leq \sum j|Q_S(j)| \leq 3\binom{m}{2} + |Q_S(4)| - \sum_{j\leq 2}|Q_S(j)|$$

$$\leq e'_S(2,2,0) + e'_S(2,0,2) + e'_S(0,2,2) + \left(3 + \frac{3\gamma}{4}\right)\binom{m}{2} - \sum_{j\leq 2}|Q_S(j)|.$$

Recall that $L_S(M) = L_{x_1}(M) \cup L_{x_2}(M) \cup L_{x_3}(M)$, so by Claim 6.3 we have

$$e'_S(2,2,0) + e'_S(2,0,2) + e'_S(0,2,2) \geq \left(\frac{1}{3} - \frac{27\gamma}{4}\right)\binom{m}{2} + \sum_{j\leq 2}|Q_S(j)|.$$

Since $e'_S(2,2,0) \leq e'_S(\cdot,\cdot,0) \leq (1/9 + 8\gamma)\binom{m}{2}$ by (21), we have

$$\sum_{j\leq 2}|Q_S(j)| \leq 31\gamma\binom{m}{2} \text{ and}$$
$$e'_S(2,2,0), e'_S(2,0,2), e'_S(0,2,2) \geq \left(\frac{1}{9} - 23\gamma\right)\binom{m}{2}.$$

Finally, we have

$$\begin{aligned}\binom{m}{2} &= \sum_{j\leq 2}|Q_S(j)| + |Q_S(3)| + |Q_S(4)| \\ &\leq \sum_{j\leq 2}|Q_S(j)| + e_S(1,1,1) + e(G'_S(\cdot,\cdot,0)) \\ &\quad + e(G'_S(\cdot,0,\cdot)) + e(G'_S(0,\cdot,\cdot)) + \frac{3\gamma}{2}\binom{m}{2} \\ &\leq \left(\frac{1}{3} + 57\gamma\right)\binom{m}{2} + e_S(1,1,1),\end{aligned}$$

so $e_S(1,1,1) \geq (2/3 - 57\gamma)\binom{m}{2}$ as required. □

Claim 6.5. *There exist two disjoint sets $S_1, S_2 \in X''$ such that G_{S_1} and G_{S_2} have at most $756\gamma\binom{m}{2}$ edges that are coloured differently or not coloured by $(2,2,0)$, $(2,0,2)$, $(0,2,2)$ nor $(1,1,1)$.*

Proof. Recall that $|X''| \geq x^3/8$. By [7], we know that there exist $x_i, x'_i \in X_i$ for $i \in [3]$ such that $\{x_1, x'_1\} \times \{x_2, x'_2\} \times \{x_3, x'_3\} \subset X''$. Let $S_1 = (x_1, x_2, x_3)$ and $S_2 = (x'_1, x'_2, x'_3)$. To prove the claim, it is sufficient to show that if S and $S' \in \{x_1, x'_1\} \times \{x_2, x'_2\} \times \{x_3, x'_3\}$ differ only in one vertex, then there are at most $252\gamma\binom{m}{2}$ edges that are coloured differently in G_S and $G_{S'}$ or not coloured by $(2,2,0)$, $(2,0,2)$, $(0,2,2)$ nor $(1,1,1)$

Without loss of generality, we may assume that $x_1 \in S$ and $x'_1 \in S'$. Note that for $E \in \binom{M}{2}$, if E is of type (a_1, a_2, a_3) with respect to S, then E is of type (a'_1, a_2, a_3) with respect to S'. Thus, if E is coloured differently in G_S and $G_{S'}$, then E is not coloured by $(2,2,0)$, $(2,0,2)$, $(0,2,2)$, $(1,1,1)$ in G_S or $G_{S'}$. By Claim 6.4, for all but at most $126\gamma\binom{m}{2}$ edges are coloured by $(2,2,0)$, $(2,0,2)$, $(0,2,2)$, $(1,1,1)$ for G_S and similarly for $G_{S'}$. Thus, the claim follows. □

Fix $S_1 = (x_1, x_2, x_3), S_2 = (x'_1, x'_2, x'_3) \in X''$ satisfying Claim 6.5. Let G be the edge-coloured subgraph of a complete graph with vertex set M induced by the edges $E \in \binom{M}{2}$ that are coloured $(1,1,1)$, $(2,2,0)$, $(2,0,2)$ and $(0,2,2)$ in both G_{S_1} and G_{S_2}. By Claim 6.4 and Claim 6.5, G has at least $(1 - 756\gamma)\binom{m}{2}$ edges. By removing at most $8\gamma\binom{m}{2}$ additional edges, we may further assume that each subgraph induced by edges of one colour

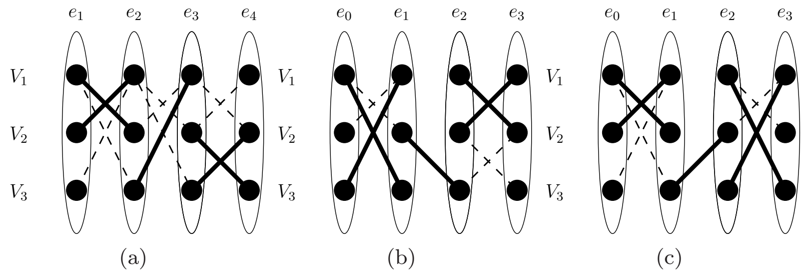


Figure 3: Diagrams for Claim 6.6

has minimum degree at least γm. Call the resulting subgraph G' and note that $e(G') \geq (1 - 764\gamma)\binom{m}{2}$,

$$e(G'(2,2,0)), e(G'(2,0,2)), e(G'(0,2,2)) \geq \left(\frac{1}{9} - 764\gamma\right)\binom{m}{2}$$

$$e(G'(1,1,1)) \geq \left(\frac{2}{3} - 764\gamma\right)\binom{m}{2}.$$

Furthermore, $|G'(2,2,0)|$, $|G'(2,0,2)|$, $|G'(0,2,2)| \geq (1/3 - 1200\gamma)\, m$. Define M_1, M_2 and M_3 to be the sets of edges $e \in M$ such that $e \in G'(0,2,2)$, $e \in G'(2,0,2)$ and $e \in G'(2,2,0)$ respectively.

Claim 6.6. *(a) M_1, M_2 and M_3 are mutually disjoint.*

(b) Let $e_1 \in M_i$ and $e_2 \in M_{i'}$ with $i \neq i'$. Then, all edges of $L_{S_1}(e_1, e_2) \cup L_{S_2}(e_1, e_2)$ incident with $v_{i,1}$ or $v_{i',2}$.

(c) Let $e_1, e_2 \in M_1$. If $e_1e_2 \in G'$, then e_1e_2 is coloured $(0,2,2)$. Similar statement holds for M_2 and M_3.

Proof. In the proof of each statement, we are going to show that there exists a matching M' of size $r+1$ in the link graphs of S_1 and S_2 spanning at most r edges of M. Moreover, $|E(M') \cap E(V_i, V_{i+1})| \leq 2$, so we can enlarge M which contradicts the maximality of M.

(a) First suppose that $M_1 \cap M_2 \neq \emptyset$. By the minimum degree of $G'(0,2,2)$ and $G'(2,0,2)$, there exists a path $e_1e_2e_3e_4$, such that $e_1e_2e_3$ is coloured $(0,2,2)$ and e_3e_4 is coloured $(2,0,2)$. However, there exists a matching of size 4 in $L_{S_i}(e_1, \ldots, e_4)$, namely $\{v_{1,1}v_{2,2}$, $v_{1,2}v_{2,1}$, $v_{1,3}v_{3,2}$, $v_{2,3}v_{3,4}$, $v_{2,4}v_{3,3}\}$ (see Figure 3 (a)).

(b) Without loss of generality, $i = 1$ and $i' = 2$. Let $e_0, e_3 \in M \backslash \{e_1, e_2\}$ such that $e_0 \neq e_3$ and e_0e_1 and e_2e_3 are of colours $(0,2,2)$ and $(2,0,2)$ respectively in G'. Suppose the claim is false, $v_{2,1}v_{3,2} \in L_{S_1}(e_1, e_2)$ say. Then, there exists a matching of size 5, namely $\{v_{1,0}v_{3,1}$, $v_{1,1}v_{3,0}$, $v_{2,1}v_{3,2}$, $v_{1,2}v_{2,3}$, $v_{1,3}v_{2,2}\}$ (see Figure 3 (b)) in $L_{S_i}(e_0, \ldots, e_3)$ for $i = 1, 2$.

(c) Let $e_0, e_3 \in M\backslash\{e_1, e_2\}$ such that $e_0 \neq e_3$ and e_0e_1 and e_2e_3 are both coloured $(0,2,2)$ in G'. Suppose that e_1e_2 is coloured by one of $\{(2,2,0),\ (2,0,2),\ (1,1,1)\}$. Without loss of generality, we may assume that $v_{2,2}v_{3,1} \in L_{S_i}(e_1, e_2)$ for $i = 1, 2$. Then, there exists a matching of size 5 in $L_{S_i}(e_0, \ldots, e_3)$ for $i = 1, 2$, namely $\{v_{1,0}v_{2,1}$, $v_{1,1}v_{2,0}$, $v_{2,2}v_{3,1}$, $v_{1,2}v_{3,3}$, $v_{1,3}v_{3,2}\}$(see Figure 3 (c)). □

For $i \in [3]$, pick $M'_i \subset M_i$ of size $(1/3 - 1200\gamma)\, m$. Let $M' = M'_1 \cup M'_2 \cup M'_3$. Now we define H' to be a 3-partite 3-graph in H induced by $V(M')$. Claim 6.3 states that $m \geq (1 - 3\sqrt{\gamma/2})n$. Hence, each partition has size $n' = 3(1/3 - 1200\gamma)m \geq (1 - \gamma'')n$. Let $W_i = M'_i \cap V_i$ and $U_i = V_i \backslash V(M'_i)$. First we show that the number of edges of type $U_1U_2U_3$ are small.

Claim 6.7. *For $u_i \in U_i$ and $i \in [3]$, we have $e(L_{u_i}(U,U,U)) \leq 800\gamma\binom{m}{2} \leq \gamma'' n'^2$. Moreover, the number of edge of type UUU is at most $2\gamma'' n'^3/3$.*

Proof. Suppose the claim is for false for some $u_1 \in U_1$, and we have $e(L_{u_1}(U,U,U)) > 800\gamma\binom{m}{2}$. Let $e_1 \in M'$ with $u_1 \in e_1$. Recall that G' has at most $764\gamma\binom{m}{2}$ missing edges. There exist $e_2, e_3 \in M'$ with $u_2 \in e_2$ and $u_3 \in e_3$ such that $e_1e_2e_3$ form a triangle in G'. Note that $e_j \in M'_{i_j}$ with $j \neq i_j$ for $j \in [3]$. In order to obtain a contradiction, it is sufficient to show that there exists a matching of size 3 in $L_{S_1}(e_1, e_2, e_3) \cap L_{S_2}(e_1, e_2, e_3)$ avoiding the vertices u_1, u_2, u_3 and so we can enlarge M together with S_1 and S_2.

First suppose that i_j are distinct. Without loss of generality we may assume that $e_1 \in M'_2$, $e_2 \in M'_3$ and $e_3 \in M'_1$. Observe that $\{v_{1,2}v_{2,1}, v_{1,3}v_{3,1}, v_{2,3}v_{3,2}\}$ is a matching of size 3. Hence, we may assume that $e_1 \in M'_2$ and $e_2, e_3 \in M'_3$. There is also a matching of size 3 in $L_{S_1}(e_1, e_2, e_3)$, namely $\{v_{1,2}v_{2,3},\ v_{1,3}v_{3,1},\ v_{2,1}v_{3,2}\}$. $\square$

Finally, we are ready to show that H' is $8\gamma''$-close to $H'_3(n'; n')$. For $u_1 \in U_1$,

$$\deg_{H'}(u_1) \geq \delta_1(H) - 2(n - n')n \geq \left(\frac{5}{9} - \gamma - 2\gamma''\right) n^2 \geq \left(\frac{5}{9} - 3\gamma''\right) {n'}^2.$$

Hence, there are $(10/27 - 2\gamma''){n'}^2$ edges of type $U_1V_2V_3$. Recall that the number of edges of type UUU is $e_{H_1}[U,U,U] \leq 2\gamma'' n'^3/3$ by Claim 6.7. Therefore,

$$e_{H'}[U_1, W_2, W_3] + e_{H'}[U_1, U_2, W_3] + e_{H'}[U_1, W_2, U_3] \geq (10/27 - 8/3\gamma''){n'}^2.$$

and similar inequalities hold when we swap the indices. Thus,

$$\begin{aligned} 3(10/27 - 8/3\gamma''){n'}^3 &\leq e_{H'}[UWW] + 2e_{H'}[UUW] \\ &\leq e_{H'}[UWW] + e_{H'}[UUW] + 4{n'}^3/9 \\ (2/3 - 8\gamma''){n'}^3 &\leq e_{H'}[UWW] + e_{H'}[UUW]. \end{aligned} \tag{22}$$

Since $e(H'_3(n'; n')) = 2n'^3/3$, H' is $8\gamma''$-close to $H'_3(n'; n')$. $\square$

7 Extremal Result

Our aim of this section is to prove the following lemma and Theorem 1.3.

Lemma 7.1. *For $0 < \epsilon < 2^{-22}$, there exists an n_0 such that the following holds. Suppose H is a 3-partite 3-graph with each class of size $n \geq n_0$. If H is ϵ-close to $H'_3(n; n)$ and*

$$\delta_1(H) > \begin{cases} \frac{4}{9}n^2 - \frac{2}{3}n & \text{if } n = 0 \pmod 3, \\ \frac{4}{9}n^2 + \frac{8}{9}n - \frac{4}{9} & \text{if } n = 1 \pmod 3, \\ \frac{4}{8}n^2 + \frac{16}{9}n - \frac{7}{9} & \text{if } n = 2 \pmod 3, \end{cases}$$

then H contains a perfect matching.

Our argument follows closely to Kühn, Osthus, and Treglown [12] (Lemma 7). Given $\alpha > 0$ and a 3-partite 3-graph H on the same partition classes as $H'(n; d_1, d_2, d_3)$, we say a vertex $v \in V(H)$ is *α-good* with respect to $H(n; d_1, d_2, d_3)$ if

$$|E(L_v^{H(n;d_1,d_2,d_3)} \backslash L_v^H)| \le \alpha n^2.$$

Otherwise v is said to be *α-bad*. Next, we show that if all vertices of H are α-good with respect to $H'(n; d_1, d_2, d_3)$ with $d_1 + d_2 + d_3 = n$, then H contains a perfect matching.

Lemma 7.2. *Let $0 < \alpha < 2^{-8}$ and let n, d_1, d_2 and d_3 be integers such that $d_1, d_2, d_3 \ge 5n/16$ and $d_1 + d_2 + d_3 = n \ge 10$. Suppose that H is a 3-partite 3-graph on the same partition classes as $H'(n; d_1, d_2, d_3)$ and every vertex of H is α-good with respect to $H'(n; d_1, d_2, d_3)$. Then H contains a perfect matching.*

Proof. Let $U_1 \cup W_1$, $U_2 \cup W_2$ and $U_3 \cup W_3$ be the partition classes of $H'(n; d_1, d_2, d_3)$ with $|W_i| = d_i$ for $i \in [3]$. Consider the largest matching M in H which consists entirely of edges of type UUW. For $i \in [3]$, define M_i to be the submatching in M consisting of edges of type UUW_i. Let $U'_i = U_i \backslash V(M)$ and $W'_i = W_i \backslash V(M)$ for $i \in [3]$. Without loss of generality, we may assume that $W'_1 \neq \emptyset$ and so $U'_2 \neq \emptyset$ and $U'_3 \neq \emptyset$. Next, we are going to show that $|M_1| \ge n/4$. Let $w_1 \in W'_1$. Since w_1 is α-good with respect to $H'(n; d_1, d_2, d_3)$ but $E(L_w) \cap (V'_2 \times V'_3) = \emptyset$, it follows that $\min\{|V'_2|, |V'_3|\} \le \sqrt{\alpha} n$, say $|V_2| \le \sqrt{\alpha} n$. Thus, $|M_1| = |V_2 \backslash V'_2| - |M_3| \ge n - d_2 - \sqrt{\alpha} n - d_3 \ge n/4$ as required.

Consider $w_1 \in W'_1$, $u_2 \in U'_2$ and $u_3 \in U'_3$. Given a pair of e_1, e_2 of distinct matching edges from M_1, we say that (e_1, e_2) is *good for $w_1 u_2 u_3$* if there are all possible edges e in H which take the form : e is of type UUW_1 and

$$|V(e) \cap V(e_1)| = |V(e) \cap V(e_2)| = |V(e) \cap \{w_1, u_2, u_3\}| = 1.$$

Note that if such pair (e_1, e_2) exists, we obtain a matching in H that is larger than M, yielding a contradiction.

Since $|M_1| \ge n/4$, we have at least $\binom{n/4}{2} \ge n^2/64$ pairs of distinct matching edges $e_1, e_2 \in M_1$. Since w_1, u_2 and u_3 are α-good with respect to $H'(n; d_1, d_2, d_3)$, there are at most $3\alpha n^2 < n^2/64$ pairs $(e_1, e_2) \in \binom{M}{2}$ such that are not good for $w_1 u_2 u_3$. So there exists a pair of edges that is good for $w_1 u_2 u_3$, a contradiction. □

Proof of Lemma 7.1. Suppose H is as in the statement of the lemma. Let $n = 3r + s + 1$ with $r \ge 0$ and $s \in [3]$. Thus, $\delta_1(H) > d_3(n, r, s)$, where $d_3(n, r, s)$ is defined in Corollary 3.4. Let $U_1 \cup W_1$, $U_2 \cup W_2$, $U_3 \cup W_3$ be the partition classes of H with

$$|W_i| = \begin{cases} r & \text{if } i = 1 \text{ and } s = 1, \\ r + 2 & \text{if } i = 3 \text{ and } s = 3, \\ r + 1 & \text{otherwise} \end{cases}$$

for $i \in [3]$. Since H is ϵ-close to $H'_3(n; n)$, all but at most $3\sqrt{\epsilon} n$ vertices in H are $\sqrt{\epsilon}$-good with respect to $H'(n; n)$. Let U_i^{bad} be the set of $\sqrt{\epsilon}$-bad vertices with respect to $H'(n; n)$ in U_i for $i \in [3]$. Define W_i^{bad} similarly.

So $|U_i^{bad}|, |W_i^{bad}| \leq 3\sqrt{\epsilon}n$ for $i \in [3]$. Let $c = \max |W_i^{bad}|$, so $c \leq 3\sqrt{\epsilon}n$. For $i \in [3]$, pick a set $\widetilde{W}_i$ such that $W_i^{bad} \subset \widetilde{W}_i \subset W_i$ and

$$|\widetilde{W}_i| = \begin{cases} c & \text{if } i = 1 \text{ and } s = 1, \\ c+2 & \text{if } i = 3 \text{ and } s = 3, \\ c+1 & \text{otherwise} \end{cases}.$$

Define $U_i^0 = U_i \cup \widetilde{W}_i$ and $W_i^0 = W_i \backslash \widetilde{W}_i$. Note that $|U_1^0| = |U_2^0| = |U_3^0| = n - r + c$.

We are going to successively remove matchings $M^1, \dots, M^5$ from H. For $j \in [5]$, let $H^j = H^{j-1} - V(M^j)$, $U_i^j = U_i^{j-1} \backslash V(M^j)$, $U^j = \bigcup_{i\in[3]} U_i^j$, $W_i^j = W_i^{j-1} \backslash V(M^j)$ and $W^j = \bigcup_{i\in[3]} W_i^j$, where $H^0 = H$. Write $m_j = |M_j|$.

Recall that $|U_i^0| = n - r + c$. Hence,

$$\delta_1(H[U^0]) \geq d_3(n, r, s) - (n^2 - |U^0||U^0|) > d_3(|U^0|, c, s).$$

Thus, there exists a matching M^1 of size $m_1 = 3c + s + 1$ in $H[U^0]$ by Corollary 3.4.

Let $H' = H_3'(n - m_1; n - m_1)$ on the vertex set $V(H^1)$. Note that if $u \in U^1$ is $\sqrt{\epsilon}$-good with respect to $H_3'(n;n)$ in H, then v is $2\sqrt{\epsilon}$-good with respect to H' in H^1 or else

$$|E(L_u^{H_3(n;n)} \backslash L_u^H)| \geq |E(L_u^{H'} \backslash L_u^{H^1})| > 2\sqrt{\epsilon}n_1^2 \geq \sqrt{\epsilon}n^2$$

contradicting u being $\sqrt{\epsilon}$-good with respect to $H_3'(n;n)$ in H. If $w \in W_1 \cap V(H^1)$ is $\sqrt{\epsilon}$-good with respect to $H_3'(n;n)$ in H, then

$$\begin{aligned} |E(L_w^{H'} \backslash L_w^{H^1})| \leq& |E(L_w^{H_3(n;n)} \backslash L_w^H)| + |\widetilde{W}_2||W_3^1| + |\widetilde{W}_3||W_2^1| \\ \leq& \sqrt{\epsilon}n^2 + 4\sqrt{\epsilon}n^2/3 + 4\sqrt{\epsilon}n^2/3 < 5\sqrt{\epsilon}n_1^2. \end{aligned}$$

Therefore, if a vertex $v \in V(H^1)$ is $5\sqrt{\epsilon}$-bad with respect to H', then $v \in \bigcup(U_i^{bad} \cup \widetilde{W}_i)$. Define $U^{1,bad}$ to be the set of such vertices, and so $|U^{1,bad}| \leq 3\sqrt{\epsilon}n$. If $U^{1,bad} = \emptyset$, then there exists a matching M_2' of size $n - m_1$ in H^1 by Lemma 7.2. Thus, $M_1 \cup M_2'$ is a perfect matching in H. So we may assume that $U^{1,bad} \neq \emptyset$.

A vertex $u \in U^{1,bad}$ is *useful* if there exist greater than $6\sqrt{\epsilon}n^2$ pairs of vertices $(u', w) \in U^1 \times W^1$ such that $uu'w$ is an edge in $H^1 \subset H$. Clearly we can greedily select a matching M_2 in H^1 of size $m_2 \leq |U^{1,bad}| \leq 3\sqrt{\epsilon}n^2$ where M_2 covers all useful vertices and consists entirely of edges of type $U^1U^1W^1$.

Consider any vertex $u \in U^{1,bad} \cap U^2$, say $u \in V_1$. Since u is not useful, it must lie in more than

$$\delta_1(H) - n(|V_2 \cap V(M)| + |V_3 \cap V(M)|) - 6\sqrt{\epsilon}n^2 - |W_2^2||W_3^3| \geq 6\sqrt{\epsilon}n^2$$

edges in $H^2[U^2]$. Since $|U^{1,bad}| \leq 3\sqrt{\epsilon}n$, once again, we can greedily select a matching M^3 in $H^2[U^2]$ of size $|U^{1,bad} \cap U^2|$ such that M^3 covers all $U^{1,bad} \cap U^2$.

Note that H^3 is a 3-partite 3-graph with each partition of size $n_3 = n - m_1 - m_2 - m_3 \geq (1 - 20\sqrt{\epsilon})n$. Let $d_i = |W_i^3|$ for $i \in [3]$. Our aim is to show that every vertex in H^3 is $6\sqrt{\epsilon}$-good with respect to $H'(n_3; d_1, d_2, d_3)$. Indeed, recall that for H^1 every vertex in $V(H^1) \backslash U^{1,bad}$ is $5\sqrt{\epsilon}$-good with respect to H', so an easy calculation shows that every vertex in H^3 is $6\sqrt{\epsilon}$-good with respect to $H'(n_3; d_1, d_2, d_3)$ as claimed. Therefore, we can

greedily find a matching M^4 of size $m_1 + m_3 \leq 13\sqrt{\epsilon}n$ edges consisting of edges of type $U^4W^4W^4$.

Finally, it is easy to see that all vertices of H^4 are $8\sqrt{\epsilon}$-good with respect to $H'(n_4; d'_1, d'_2, d'_3)$, where $d'_i = |W_i^4|$ for $i \in [3]$. Note that $d'_i \geq 5n_4/16$ and

$$d'_1 + d'_2 + d'_3 = n - m_2 - 2m_4 = n - \sum_{i\in[4]} m_i = n_4.$$

Thus, we can apply Lemma 7.2 to obtain a perfect matching M^5 in H^4, so $M^1 \cup \cdots \cup M^5$ is a perfect matching in H. □

Finally, we are ready to prove Theorem 1.3.

Proof of Theorem 1.3. Let $\gamma > 0$ be sufficiently small and n_0 be sufficiently large. Set $\rho = 6\gamma^6$ and $\gamma'' = 3(1200\gamma + \sqrt{\gamma/2})$. Since γ is small, H satisfies the hypothesis of Lemma 4.2 with $k = 3$ and $l = 1$. Let M be the matching given by Lemma 4.2 and so $|M| \leq \gamma^3 n$. Let $H' = H \backslash V(M)$. Note that

$$\delta_1(H') \geq \delta_1(H) - 2\gamma^3 n^2 \geq \left(\frac{5}{9} - \gamma\right) n'^2$$

where $n' = n - |M|$. First suppose that there exists a matching M' in H' covering all but at most $\rho n' \leq 6\gamma^6 n$ vertices in each partition class. Let $W = V(H)\backslash V(M \cup M')$. By Lemma 4.2, there exists a matching M'' covering exactly the vertices of $V(M) \cup W$. Thus, M' and M'' form a perfect matching.

Therefore, we may assume that there is no matching in H' of size $(1-\rho)n'$. By Lemma 6.1, there exists a subgraph H'' in H' such that H'' is $8\gamma''$-close to $H'_3(n''; n'')$, where $n'' \geq (1-\gamma'')n' \geq (1-2\gamma'')n$ and $3|n''$. Note that

$$\begin{aligned} |E(H'_3(n;n))\backslash E(H)| \leq& |E(H'_3(n;n))\backslash E(H'')| \\ \leq& 8\gamma'' n''^3 + 3(n-n'')n^2 \leq 14\gamma'' n^3, \end{aligned} \tag{23}$$

so H is $14\gamma''$-close to $H'_3(n;n)$. Therefore H has a perfect matching by Lemma 7.1. □

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